

Kinematics of ultra-high energy particle collisions near black holes in the magnetic field

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There are different versions of collisions of two particles near black holes with unbound energy $E_{c.m.}$ in the centre of mass frame. The so-called BSW effect arises when a slow fine-tuned "critical" particle hits a rapid "usual" one. We discuss a scenario of collision in the strong magnetic field for which explanation turns out to be different. Both particles are rapid but the nonzero angle between their velocities (which are both close to c , the speed of light) results in a relative velocity close to c and, hence, big $E_{c.m.}$.

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I. INTRODUCTION

Several years ago, an interesting observation was made that the collision between two particles near the extremal Kerr metrics can lead to the unbound growth of the energy in their centre of mass $E_{c.m.}$ [1] although individual Killing energies E_1 and E_2 are finite. This effect is now called the Bañados - Silk - West (BSW) effect. This result came as a surprise but later it was shown that the BSW effect reveals itself for quite generic black holes and has a simple kinematic explanation [2]. The BSW effect occurs if one particle is so-called critical (has fine-tuned energy and angular momentum or electric charge) while the second particle is "usual" (not fine-tuned). Then, in the stationary or static reference frame, a usual

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particle moves near the horizon with a speed approaching that of light, whereas the velocity of the critical one is separated from it. Then, a very rapid particle hits a slow target that results in a large Lorentz factor of relative motion, hence large $E_{c.m.}$.

Meanwhile, another kind of high-energy collisions was proposed that requires the magnetic field [3]. If the black hole is weakly magnetized but the charge-to-mass ratio q/m of a particle is big enough, the effect happens even in the simplest Schwarzschild background, so one can neglect

the backreaction of the magnetic field on the metric. In the absence of a black hole rotation, there are no special relations between the energy and the angular momentum, so there is no direct analogue of the critical particle typical of the Kerr metric. However, kinematic explanation of the effect considered in [3] is very similar. One particle moves on the innermost stable circular orbit (ISCO) that lies close to the horizon for a sufficiently large magnetic field, the second one being arbitrary, so again we have separation to two classes of particles.

Quite recently, another version of collisions in the magnetic field, not connected with ISCO, was proposed in [4]. It was shown that there exists a sharp region near the horizon where collisions in the magnetic field strong enough lead to unbound $E_{c.m.}$ for *any* two particles that lends some universality to the effect.

Then, it seems that we have a paradox since we have high $E_{c.m.}$ without separation to critical and usual particles. Meanwhile, it was found earlier that collision between two usual particles gives rise to modest $E_{c.m.}$ [2]. The aim of the present work is to show that the effect under discussion in the magnetic field has another underlying kinematic reason as compared to the original BSW effect. We give below simple kinematic explanation in the same sense as Ref. [2] gave explanation to the BSW effect. Throughout the paper we use units in which fundamental constants are $G = c = 1$.

II. METRIC AND EQUATIONS OF MOTION

Let us consider the static metric

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{A} + R^2 d\phi^2 + g_\theta d\theta^2, \quad (1)$$

where the metric coefficients do not depend on t and ϕ . The horizon lies at $r = r_+$ and corresponds to $N = 0$. We also assume that there is an electromagnetic field with the four-vector A^μ

$$A^\phi = \frac{B}{2}, \quad (2)$$

all other components $A^\mu = 0$.

In vacuum, this is an exact solution with $B = \text{const}$ [5]. We consider configuration with matter (in this sense a black hole is "dirty"), so in general B may depend on r and θ . We consider motion constrained within the equatorial plane, so $\theta = \frac{\pi}{2}$. Redefining the radial coordinate $r \rightarrow \rho$, we can always achieve that

$$A = N^2 \quad (3)$$

within this plane. Then, equations of motion read

$$\dot{t} = \frac{E}{N^2 m}, \quad (4)$$

$$\dot{\phi} = \frac{\beta}{R}, \quad (5)$$

$$m\dot{r} = \varepsilon Z, \quad (6)$$

$$\beta = \frac{L}{mR} - \frac{qBR}{2m}, \quad (7)$$

$$Z = \sqrt{E^2 - m^2 N^2 (1 + \beta^2)}, \quad (8)$$

$\varepsilon = -1$ if the particle moves towards the horizon and $\varepsilon = +1$ in the opposite case. Here, m is the particle's mass, q is its electric charge.

To describe kinematic properties of particles, it is convenient to introduce a tangent locally flat space with the use of tetrads. This will enable us to carry out close analogy with special relativity. Then, the components of the local three-velocity are equal to

$$V_{(a)} = V^{(a)} = -\frac{u^\mu h_{(a)\mu}}{u^\mu h_{(0)\mu}}. \quad (9)$$

Here, u^μ is the four-velocity, $h_{(a)\mu}$ is the tetrad, $\mu = 0, 1, 2, 3$, $(a) = 1, 2, 3$. Then, denoting coordinates x^μ as $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, it is natural to choose the tetrad in the form $h_{(0)\mu} = -N(1, 0, 0, 0)$, $h_{(1)\mu} = N^{-1}(0, 1, 0, 0)$, $h_{(2)\mu} = \sqrt{g_{\theta\theta}}(0, 0, 1, 0)$, $h_{(3)\mu} = R(0, 0, 0, 1)$. Then, it follows from (4) - (8) that

$$V^{(1)} = \sqrt{1 - \frac{N^2}{\alpha^2}(1 + \beta^2)} \quad (10)$$

$$V^{(3)} = \frac{\beta N}{\alpha}, \quad (11)$$

where

$$\alpha = \frac{E}{m}. \quad (12)$$

The component $V^{(2)} = 0$ since a particle moves within the equatorial plane. From (10), (11) it follows that

$$E = m\gamma_0 = \frac{mN}{\sqrt{1-V^2}}, \quad (13)$$

where $V^2 = (V^{(1)})^2 + (V^{(3)})^2$, γ_0 has the meaning of the Lorentz gamma factor for a given particle.

III. PARTICLE COLLISIONS

Let two particle collide at some point where $r = r_c$. In this point, one can define the energy in the centre of mass frame according to

$$E_{c.m.} = -P^\mu P_\mu = m_1^2 + m_2^2 + 2m_1m_2\gamma, \quad (14)$$

$$P^\mu = m_1u_1^\mu + m_2u_2^\mu, \quad (15)$$

the Lorentz factor of relative motion (not to be confused with the factor γ_0 for each individual particle)

$$\gamma = -u_{1\mu}u_2^\mu, \quad (16)$$

indices 1 and 2 label the particles. Then, using (4) - (8), one finds

$$\gamma = \frac{E_1E_2 - \varepsilon_1\varepsilon_2Z_1Z_2}{N^2m_1m_2} - \beta_1\beta_2. \quad (17)$$

Let us define the function $s(r) = N\beta$, $s_i \equiv s_i(r_c)$ for particle i . We assume that at least for one of two particles

$$s_i \sim 1. \quad (18)$$

This means that the large β compensates the small factor N in the point of collision near the horizon. Then, the gamma factor reads [4]

$$\gamma \approx \frac{F}{N^2(r_c)}, \quad F = \alpha_1\alpha_2 - \sqrt{\alpha_1^2 - s_1^2}\sqrt{\alpha_2^2 - s_2^2} - s_1s_2, \quad (19)$$

it grows unbound near the horizon.

If near the horizon we neglect a small difference between $\beta(r)$ and $\beta(r_+)$,

$$s_i(r) \approx s_i \frac{N(r)}{N(r_c)}. \quad (20)$$

Thus $s_i(r_+) = 0$. The unbound γ is possible in the strip near the horizon where $s_i \leq \alpha_i$ or, equivalently,

$$0 < N_c \leq \frac{E_i}{m_i \beta(r_+)}. \quad (21)$$

In other words, for any value of magnetic field large enough, so that $|\beta(r_+)| \gg 1$, there exist a narrow strip around horizon within which $\gamma \sim N_c^{-2} \sim B^2$ is large.

The mechanism under discussion works both for the nonextremal ($N^2 \sim (r - r_+)$) and extremal ($N^2 \sim (r - r_+)^2$) cases. In other words, we require N to be small in the point of collision near the horizon but do not specify the rate with which it vanishes.

It is worth noting that for collisions near ISCO, self-gravitation of very strong magnetic fields ($BM \gtrsim 1$, where M is a black hole mass) restricts the growth of $E_{c.m.}$ (see Sec. XII of [6]) since the location of ISCO fails to be close to the horizon. By contrast, in the case under discussion the effect is not tight to ISCO, so high energies are achievable as long as eq. (18) is satisfied in the strong magnetic field near the horizon.

These features give more universality to the effect under discussion.

IV. PROPERTIES OF VELOCITIES

Properties of the gamma factor can be understood in terms of the relative velocity w of two particles:

$$\gamma = \frac{1}{\sqrt{1 - w^2}}, \quad (22)$$

The unbound growth of γ (hence, the BSW effect) happens when $w \rightarrow 1$. If in the flat space-time particles have the three-velocities $\vec{V}_1 = V_1 \vec{n}_1$ and $\vec{V}_2 = V_2 \vec{n}_2$,

$$w^2 = 1 - \frac{(1 - V_1^2)(1 - V_2^2)}{[1 - V_1 V_2 (\vec{n}_1 \vec{n}_2)]^2}, \quad (23)$$

see for details problem 1.3. in [7]. Possible cases in which w approaches 1 (speed of light in our units) can be classified depending on V_1 , V_2 and the angle between them, determined by $(\vec{n}_1 \vec{n}_2)$. For the standard BSW effect, the only relevant case occurs when $V_1 < 1$, $V_2 \rightarrow 1$, $(\vec{n}_1 \vec{n}_2)$ is arbitrary (case "a" according to classification given in Sec. III of Ref. [2]). Then, it is clear from (23) that indeed $w \rightarrow 1$.

However, now eq. (13) tells us that for any finite E , the velocity $V \rightarrow 1$ on the horizon, so the critical particle in the sense of [2] is impossible. It was possible for rotating black holes since for them the left hand side of eq. (13) is to be replaced with $E - \omega L$, where the metric coefficient ω is responsible for rotation, so for the critical particle $E = \omega_H L$, where ω_H is the angular velocity of a black hole. But now $\omega_H = 0$, $E > 0$. Happily, there is one more case that ensures the desirable limit of w . Let $V_1 \rightarrow 1$, $V_2 \rightarrow 1$ in such a way that

$$V_i = 1 - A_i \delta \quad (24)$$

where A_i ($i = 1, 2$) are constants, $\delta \ll 1$,

$$(\vec{n}_1 \vec{n}_2) \neq 1. \quad (25)$$

This is case b1 from [2]. Then, it follows from (23) that

$$w^2 \approx 1 - \frac{4A_1 A_2 \delta^2}{[1 - (\vec{n}_1 \vec{n}_2)]^2}, \quad (26)$$

so we have $w \rightarrow 1$ again.

Now we will see that for collisions in the magnetic field under discussion it is this case which is realized. To demonstrate this, we must check two conditions - (24) and (25). The validity of (24) follows immediately from (13): for both particles $V_1 = 1 - O(N^2)$, so they approach 1 with the same rate as required.

Now, we must check (25). It is instructive to compare two cases - (i) the magnetic field is absent or at least finite, so $\beta(r_+) \lesssim 1$ (ii) condition (18) is satisfied, so $\beta(r_+) \gg 1$.

In case (i) typical of the standard BSW effect [1], [2], in the horizon limit $N \rightarrow 0$ we obtain that $V^{(1)} \rightarrow 1$, $V^{(3)} \rightarrow 0$. It means that when a particle crosses the horizon, its velocity is pointed always in the same direction. Namely, it is perpendicular to the horizon for any particle, so that $(\vec{n}_1 \vec{n}_2) \rightarrow 1$: rains falls vertically on a black hole - see p. F-17 of [8]. Therefore, $(\vec{n}_1 \vec{n}_2) = 1$, so (25) is violated.

In case (ii), it follows from (18), (10), (11) that

$$V^{(1)}(r_c) \approx \varepsilon \sqrt{1 - \frac{s^2}{\alpha^2}}, \quad (27)$$

$$V^{(3)}(r_c) \approx \frac{s}{\alpha}. \quad (28)$$

Thus the angle with which they approach the horizon can be arbitrary and different for different particles, so that (25) is satisfied. More precisely, one can find from (10), (11) that for $\varepsilon_1 = \varepsilon_2 = -1$,

$$(\vec{n}_1 \vec{n}_2) = \frac{\sqrt{\alpha_1^2 - s_1^2} \sqrt{\alpha_2^2 - s_2^2} + s_1 s_2}{\alpha_1 \alpha_2}. \quad (29)$$

From (13), we have

$$1 - V_i^2 = \frac{N^2}{\alpha_i^2}. \quad (30)$$

Then, using (22) and (23), one finds just the expression (19) that was derived in [4] from the equations of motion.

It is worth noting that this is valid for the point of collision, where the product $N\beta$ is finite according to (18). However, for any large but finite magnetic field (so β is finite as well), this product vanishes on the horizon itself, so again $(\vec{n}_1 \vec{n}_2) \rightarrow 1$, $V \rightarrow 1$ for both particles. Although, according to (21), in the close vicinity of the, say, nonextremal horizon $r_c - r \sim b^{-2}$, "rain does not fall vertically", its direction is restored when $r - r_+ \ll r_c - r$, so $s(r) \ll 1$.

In accordance with the general rule that collision of two usual particles does not produce unbound $E_{c.m.}$ [2], the Lorentz factor γ becomes finite. Indeed, it follows from (17) that $\lim_{r \rightarrow r_+} \gamma \sim \beta^2(r_+)$. However, if the magnetic field is strong enough, γ can be made as large as one likes.

V. SUMMARY

Thus the original BSW effect and high-energy collision in the magnetic field in the immediate vicinity of the horizon [4] have two complimentary kinematic explanations. They correspond to cases (a) and (b) of eqs. 19 - 20 in [2]. Namely, in case (a) high $E_{c.m.}$ are achieved when one particle is critical (fine-tuned, slow) and the second particle is usual (not fine-tuned, rapid). For scenarios of the type considered in [4], both particles are usual and rapid, big $E_{c.m.}$ are obtained since their velocities are non-parallel to each other. Thus the kinematic explanations of both processes (without the magnetic field and in the strong magnetic field) prove to be two different versions of the same underlying picture.

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